Divisibility Rules

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A collection of divisibility rules for small primes, complete with examples and proofs. These divisibility rules are not my invention, most have been known for hundreds of years. In many cases there are more than one divisibility rule, I have attempted to include the simplest one, after all if a rule is more work then dividing, it isn't much practical use. I have included more than one proof for several rules, ordered by difficulty. Generally the first proof is most suitable for students first learning the rule, while other proofs are more rigorous or use more advanced topics.

divisibility by 2

A number is divisible by 2 if the last digit is 0,2,4,6, or 8

Example: 24 ends in a 4, so it is divisible by 2. 395 ends with a 5, so it is not divisible by 2.

Proof. 2 is divisible by 2. So is 4, because $2 \times 2 = 4$. $2 \times 3 = 6$, $2 \times 4 = 8$, $2 \times 5 = 10$ and $2 \times 6 = 12$ Look at the last digits, 2, 4, 6, 8, 0, then 2 again. The pattern continues, so multiples of 2 must end with 0,2,4,6, or 8. □

Alternate Proof. If a number is divisible by 2, it can be written as $2 \times x$ where x is some positive integer. Now suppose that you multiply a number by 2. You first multiply the digit in the ones place by two (carry if necessary) then multiply the number in the tens place, and so on. In the ones place is the ones digit of the first multiplication, the original ones place times two. Now consider every digit times two:

digit	digit×2
0	0
1	2
2	4
3	6
4	8
5	10
6	12
7	14
8	16
9	18

Note that in each case the ones digit of the result ends in 0,2,4,6, or 8. Thus every number, when multipled by 2, will have one of these digits in the ones place. Thus, every number with one of these digits in the ones place (ending with 0,2,4,6, or 8) must be 2× some number, which means it is divisible by 2. □

divisibility by 3

Add all the digits up, if the sum is divisible by 3, then the number is as well

Example: Consider 15 1+5=6 6 is divisible by 3 so 15 is divisible by 3. 124 1+2+4=7 7 is not divisible by 3, so 124 is not divisible by 3.

Proof. All positive integers z may be written as $z = a_0 + 10a_1 + 100a_2 + 1000a_3 + \ldots$ where $a_i \in \{0, 1, \ldots 9\}$. Typically we write numbers as $a_n \ldots a_3a_2a_1a_0$. Now consider this number mod 9. It may be written as $a_0 + (9+1)a_1 + (9+1)^2a_2 + \ldots$ Observe that mod 9 this is equivalent to $a_0 + a_1 + a_2 + \ldots$. So, mod 9, a number and the sum of its digits are equivalent. If $k \equiv z \pmod 9$, there is some m such that $z = k + 9m$. Consider this expression mod 3. But $9 = 3 \times 3$ so we need only consider k mod 3. The result follows. □

divisibility by 5

A number is divisible by 5 if the last digit is 0 or 5 Example 10, 15, 25, and 50 are divisible by 5 because they end in 0 or 5, 13, 22, 6 are not because they don't end in a 0 or 5.

Proof. This proof is similar to the proof for divisibility by 2, the product of any single digit and 5 ends in either a 0 or 5.

digit	digit×5
0	0
1	5
2	10
3	15
4	20
5	25
6	30
7	35
8	40
9	45

Thus the digit in the ones place must be either a 0 or 5. □

divisibility by 7

Take the last digit, double it, and subtract that value from the number formed by the remaining digits. If the result is divisible by 7, so is the number.

Example: Consider 154. The last digit is 4, double that, getting 8, and subtract 8 from the number formed by the remaining digits, in this case 15. $15 - 8 = 7$. 7 is divisible by 7, so 154 is divisible by 7. Consider 218. The last digit is 8, double that, getting 16, and subtract that from the remaining digits, in this case 21. $21 - 16 = 5$. 5 is not divisible by 7, so neither is 218.

Proof. Note that any integer z can be written as $10a + b$. Now consider the quantity $10(a - 2b)$ This is equal to $z \mod 7$, because $10 \times 2 \equiv -1 \mod 7$. Since 10 is not divisible by 7, $10(a-2b)$ is divisible by 7 iff $a-2b$ is. The result follows. □

divisibility by 11

take the first number, subtract the second, add the third, and so on. If the absolute value of this result is 0 or divisible by 11, the number is divisible by 11.

Example: Consider 6248. $8-4+2-6 = 0$ so 6248 is divisible by 11. Consider 1234567. $7-6+5-4+3-2+1 = 4$ so it is not divisible by 11.

Proof. Simmilar to the proof for divisibility by 3. All positive integers z may be written as $z = a_0 + 10a_1 + 100a_2 + 1000a_3 + \dots$ where $a_i \in \{0, 1, \dots 9\}$. Typically we write numbers as $a_n \dots a_3a_2a_1a_0$. Now consider this number mod 11. It may be written as $a_0 + (11-1)a_1 + (11-1)^2a_2 + \dots$ which is equivalent to $a_0 - a_1 + a_2 + \dots$ and the result follows. □

Divisibility by composite numbers

A list of divisibility rules for composite numbers without proof. In most cases the proof is trivial.

number	divisibility rule
4	A number is divisible by 4 if the number composed of the last two digits are divisible by 4.
6	A number is divisible by 6 if it is divisible by both 2 and 3. In other words, it ends in a 0,2,4,6, or 8 *and* the sum of the digits is divisible by 3.
8	A number is divisible by 8 if the number composed of the last three digits are divisible by 8.
9	A number is divisible by 9 if the sum of the digits is divisible by 9.
10	A number is divisible by 10 if it is divisible by 2 and 5. In other words, it must end in a 0.